

# PRYM-TJURIN CONSTRUCTIONS ON CUBIC HYPERSURFACES

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**ABSTRACT.** In this paper, we give a Prym-Tjurin construction for the cohomology and Chow groups for a cubic hypersurface. On the space of lines meeting a given rational curve, there is the incidence correspondence. This correspondence induces an action on both primitive cohomology and Chow groups. We first show that this action satisfies a quadratic equation. Then the Abel-Jacobi mapping induces an isomorphism between the primitive cohomology of the cubic hypersurface and the Prym-Tjurin part of the above action. This also holds for Chow groups with rational coefficients. All the constructions are based on a natural relation among topological (resp. algebraic) cycles on  $X$  modulo homological (resp. rational) equivalence.

## 1. INTRODUCTION

Algebraic cycles on a cubic hypersurface has been serving as an interesting but nontrivial case of study of Chow groups. In [Sh], the author obtains natural relations among 1-cycles and gives some applications of those relations. This paper is a continued work of [Sh] and generalization of the results to higher dimensional cycles on cubic hypersurfaces. Throughout of the the paper, we will work over the field  $\mathbb{C}$  of complex numbers.

Let  $X \subset \mathbb{P}^{n+1}$  be a smooth cubic hypersurface of dimension  $n \geq 3$ . We first establish Theorem 2.2 which gives a natural relation among cycles on  $X$ . To get an idea what such a relation is, we first fix a smooth curve  $C \subset X$  of degree  $e$ . Let  $T \subset X$  be a closed subvariety of dimension  $r$ . Let  $T' \subset X$  be the subvariety swept out by lines on  $X$  that meet both  $C$  and  $T$ . When  $C$  and  $T$  are in general position, then  $T'$  has expected dimension which is equal to  $r$ . If  $r \geq 2$ , then the relation we get is

$$2eT + T' = ah^{n-r}, \quad \text{in } \text{CH}_r(X)$$

for some integer  $a > 0$ , where  $h$  is the class of hyperplane section of  $X$ . If  $r = 1$ , the relation we get is

$$2eT + T' + 2\deg(T)C = bh^{n-1}, \quad \text{in } \text{CH}_1(X)$$

for some integer  $b > 0$ . This was first proved in [Sh].

Let  $F$  be the Fano scheme of lines on  $X$ . Let  $p : P \rightarrow F$  be the total family of lines on  $X$  and  $q : P \rightarrow X$  be the natural morphism. Then we have the natural cylinder homomorphism  $\Psi = q_*p^*$  and its transpose  $\Phi = p_*q^*$ . These are defined on both cohomology and Chow groups. The following theorem is a more concise expression of the above relations.

**Theorem 1.1.** *Let  $\gamma \in H_n(X, \mathbb{Z})$ ,  $\mathbf{a} \in \text{CH}_1(X)$  and  $\mathbf{b} \in \text{CH}_l(X)$ . Then the following are true*

- (i)  *$2\deg(\mathbf{a})\gamma + \Psi(\Phi(\gamma) \cdot \Phi([\mathbf{a}]))$  is a homology class that comes from  $\mathbb{P}^{n+1}$ .*
- (ii) *If  $n - 2 \geq l \geq 2$ , then  $2\deg(\mathbf{a})\mathbf{b} + \Psi(\Phi(\mathbf{a}) \cdot \Phi(\mathbf{b}))$  is a cycle class that comes from  $\mathbb{P}^{n+1}$ .*
- (iii) *If  $l = 1$ , then  $2\deg(\mathbf{a})\mathbf{b} + 2\deg(\mathbf{b})\mathbf{a} + \Psi(\Phi(\mathbf{a}) \cdot \Phi(\mathbf{b}))$  is a cycle class that comes from  $\mathbb{P}^{n+1}$ .*

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**Remark 1.2.** If  $n \geq 5$ , then  $\mathrm{CH}_1(X) \cong \mathbb{Z}$ . In this case,  $\mathfrak{a} \in \mathrm{CH}_1(X)$  is the same as  $\deg(\mathfrak{a})[l]$  where  $[l]$  is the class of a line. For fixed  $n$ , the groups  $\mathrm{CH}_l(X)$  are expected to be trivial (isomorphic to  $\mathbb{Z}$ ) when  $l$  small or big enough. Hence, the above relations are interesting when  $l$  is in the middle range.

With these relations, we will realize the Hodge structure and Chow groups of  $X$  as Prym-Tjurin constructions.

A Prym variety is constructed from a curve together with an involution and they form a larger class of principally polarized abelian varieties (p.p.a.v.) than Jacobian of curves. Mumford gives this a modern treatment in [Mu]. He also uses Prym varieties to characterize the intermediate Jacobian of a cubic threefold, see the appendix of [CG]. In [Tj], Tjurin generalizes this idea by replacing the involution with a correspondence that satisfies a quadratic equation. This gives what we now call Prym-Tjurin varieties. This was further developed and completed by S. Bloch and J.P. Murre, see [BM]. Welters has proved that all p.p.a.v.'s can be realized as Prym-Tjurin varieties, see [We]. Roughly speaking, this means that weight one Hodge structures can always be realized via Prym-Tjurin constructions on curves. We naturally ask whether similar constructions can be done for higher weight Hodge structures. The work of Lewis in [Lew] sheds some lights in this direction. Izadi gives a Prym construction for the cohomology of cubic hypersurfaces in [Iz].

In this article, all cohomology groups are modulo torsion.

**Definition 1.3.** Let  $\Lambda$  be an abelian group and  $\sigma : \Lambda \rightarrow \Lambda$  be an endomorphism. Assume that  $\sigma$  satisfies the quadratic equation  $(\sigma - 1)(\sigma + q - 1) = 0$ . Then the *Prym-Tjurin part* of  $\Lambda$  is defined as

$$P(\Lambda, \sigma) = \mathrm{Im}(\sigma - 1)$$

**Remark 1.4.** In many cases when  $\Lambda$  carries some extra structure such as Hodge structure, the homomorphism  $\sigma$  is usually compatible with that extra structure and  $P(\Lambda, \sigma)$  carries an induced such structure. One very interesting case is when  $\Lambda = H^*(Y, \mathbb{Z})$  together with the natural Hodge structure and  $\sigma$  is an action that is induced by some correspondence  $\Gamma \in \mathrm{CH}^{\dim Y}(Y \times Y)$ .

Let  $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$  be a smooth cubic hypersurface as above. We start with a general rational curve  $C \subset X$  of degree  $e \geq 2$ . Let  $S_C$  be a natural resolution of the space of lines on  $X$  that meet  $C$ . We will construct  $S_C$  explicitly. Let  $F = F(X)$  be the Fano scheme of lines on  $X$ ,  $p : P \rightarrow F$  the total family and  $q : P \rightarrow X$  the natural morphism. Then  $S_C = q^{-1}(C)$ . In particular, we have a natural morphism  $q : S_C \rightarrow C$ . By considering the family  $p_C : P_C \rightarrow S_C$  of lines parameterized by  $S_C$  with the natural cycle map  $q_C : P_C \rightarrow X$ , we get a natural cylinder homomorphisms  $\Psi_C = (q_C)_*(p_C)^* : H^{n-2}(S_C, \mathbb{Z}) \rightarrow H^n(X, \mathbb{Z})$  and  $\Psi_C = (q_C)_*(p_C)^* : \mathrm{CH}_r(S_C) \rightarrow \mathrm{CH}_{r+1}(X)$ . Similarly we have its transpose  $\Phi_C = (p_C)_*(q_C)^* : H^n(X, \mathbb{Z}) \rightarrow H^{n-2}(S_C, \mathbb{Z})$  and  $\Phi_C = (p_C)_*(q_C)^* : \mathrm{CH}_{r+1}(X) \rightarrow \mathrm{CH}_r(S_C)$ . On  $S_C$  we have the naturally defined incidence correspondence which induces an action  $\sigma$  on both cohomology and Chow groups. Note that there is a natural morphism from  $S_C$  to the Grassmannian  $G = G(2, n+2)$ . The Plücker embedding of  $G$  induces an ample class  $g$  on  $S_C$ . On  $S_C$ , we have another divisor class  $g' = q^*(pt)$  for some  $pt \in C$ . Let  $\mathbb{Q}[g, g'] \subset H^*(S_C, \mathbb{Q})$  be the subring generated by  $g$  and  $g'$ . We define the primitive cohomology (or cycle) classes on  $S_C$  to be those which are orthogonal to the classes in  $\mathbb{Q}[g, g']$ . The primitive cohomology is denoted by  $H^*(S_C, \mathbb{Z})^\circ$  and the primitive Chow group is denoted by  $\mathrm{CH}_*(S_C)^\circ$ . Our main result is the following

**Theorem 1.5.** *Let  $C \subset X$  be a general rational curve of degree  $e \geq 2$  as above. Let  $\sigma$  be the action of the incidence correspondence on either primitive cohomology or primitive Chow group. Then the following are true.*

(1) *The action  $\sigma$  satisfies the following quadratic relation*

$$(\sigma - 1)(\sigma + 2e - 1) = 0$$

(2) *The map  $\Phi_C$  induces an isomorphism of Hodge structures*

$$\Phi_C : H^n(X, \mathbb{Z})_{\text{prim}} \rightarrow P(H^{n-2}(S_C, \mathbb{Z})^\circ, \sigma)(-1)$$

*where the  $(-1)$  on the right hand side means shifting of degree by  $(1, 1)$ . The intersection forms are related by the following identity*

$$\Phi_C(\alpha) \cdot \Phi_C(\beta) = -2e \alpha \cdot \beta$$

(3) *The map  $\Phi_C$  induces an isomorphism*

$$\Phi_C : A_i(X)_{\mathbb{Q}} \rightarrow P(\text{CH}_{i-1}(S_C)_{\mathbb{Q}}^\circ, \sigma)$$

This is proved in section 4 (Theorem 4.2). In [Iz], Izadi has proved a variant of statement (2) for  $C$  being a line. In [Sh], the author has proved the above theorem for cubic threefolds where (3) holds true for integral coefficients. Besides the natural relations stated at the very beginning of this section, another ingredient to carry out the Prym-Tjurin construction is the surjectivity of  $\Psi_C$  on primitive cohomologies.

**Theorem 1.6.** *The natural homomorphism*

$$\Psi_C : H^{n-2}(S_C, \mathbb{Z})^\circ \rightarrow H^n(X, \mathbb{Z})_{\text{prim}}$$

*between primitive cohomologies is surjective.*

This is proved in Section 5 using the Clemens-Letizia method (see [Cl] and [Le]).

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Notations:

$X \subset \mathbb{P}^{n+1}$ , smooth cubic hypersurface of dimension  $n \geq 3$  with  $h$  being the hyperplane;

$G = G(2, n+2)$  is the Grassmannian of lines in  $\mathbb{P}^{n+1}$ ;

$F = F(X) \subset G$ , Fano scheme of lines on  $X$ , smooth of dimension  $2n-4$ ;

$P = P(X)$ , the universal family of lines on  $X$ , namely we have the following diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ p \downarrow & & \\ F & & \end{array}$$

$I \subset F \times F$ , the incidence correspondence, i.e.  $I = \{(l_1, l_2) : l_1 \text{ meets } l_2\}$ ; Note that  $I$  has codimension  $n-2$  in  $F \times F$ ;

$\Phi = p_* q^*$ , acting either on cohomology or Chow groups;

$\Psi = q_* p^*$ , acting either on cohomology or Chow groups;

$g$  is the polarization on  $F$  that comes from the Plücker embedding,  $g = \Phi(h^2)$ ;

$F_x \subset F$  is the subscheme of lines passing through  $x \in X$ ; it is a  $(2, 3)$  complete intersection in  $\mathbb{P}(T_{X,x})$  and smooth for general  $x$ .

$F_C \subset F$  is the subscheme parameterizing a lines meeting a curve  $C \subset X$ ;  
 $S_C = q^{-1}(C)$ , note that there is a natural morphism  $S_C \rightarrow F_C \subset F$ ;  
 $P_C = P|_{S_C}$ ,  $q_C = q|_{P_C}$ ,  $p_C = p|_{P_C}$ ;  
 $\Phi_C = (p_C)_*(q_C)^*$ , acting either on cohomology or Chow groups;  
 $\Psi_C = (q_C)_*(p_C)^*$ , acting either on cohomology or Chow groups;  
 $D_C \subset X$ , the divisor swept out by all the lines meeting  $C$ ;  $D_C$  is linearly equivalent to  $5 \deg(C)h$ ;

All cohomology or homology groups considered in this article are modulo torsion; given a polarization  $H$  on  $Y$ , we use  $H^*(Y, \mathbb{Z})_{\text{prim}}$  to denote the primitive cohomology;

Let  $(Y, H)$  be polarized, we define  $A_*(Y) \subset CH_*(Y)$  to be the subgroup of degree 0 (with respect to  $H$ ) cycles;

For a vector bundle  $\mathcal{E}$  on  $Y$ , we use  $\mathbb{P}(\mathcal{E})$  to denote the geometric projectivization which parameterizes all 1-dimensional linear subspaces of  $\mathcal{E}$ ;

## 2. THE FUNDAMENTAL RELATIONS

In this section we will establish a basic relation among algebraic/topological cycles on  $X$ . To do that, we need the following lemma which says that the space of lines on  $X$  meeting a given curve has the expected dimension.

**Lemma 2.1.** *Let  $C \subset X$  be a smooth curve on  $X$ . Then  $F_C$  has the expected dimension  $n - 2$ .*

*Proof.* It is known that  $\dim F_x = n - 3$  for a general point  $x \in X$ . There are at most finitely many points  $x_i$  such that  $\dim F_{x_i} = n - 2$ . Thus the Lemma follows. See [CS], Corollary 2.2.  $\square$

**Theorem 2.2.** *Let  $C \subset X$  be a smooth curve of degree  $e$ . Then the following are true.*

(1) *Let  $\gamma$  be a topological  $n$ -dimensional cycle on  $X(\mathbb{C})$ . Then*

$$2e[\gamma] + \Psi(\Phi([\gamma]) \cdot F_C)$$

*is a homology class that comes from  $\mathbb{P}^{n+1}$  which only depends on the degrees of  $\gamma$  and  $C$ .*

(2) *Let  $\gamma$  be an algebraic cycle of dimension between 2 and  $n - 2$  on  $X$ . Then*

$$2e\gamma + \Psi(\Phi(\gamma) \cdot F_C)$$

*is a cycle class (modulo rational equivalence) that comes from  $\mathbb{P}^{n+1}$  which only depends on the degrees of  $\gamma$  and  $C$ .*

(3) *Let  $\gamma$  be an algebraic cycle of dimension 1 on  $X$ . Then*

$$2e\gamma + \Psi(\Phi(\gamma) \cdot F_C) + 2 \deg(\gamma)C$$

*is a cycle class (modulo rational equivalence) that comes from  $\mathbb{P}^{n+1}$  which only depends on the degrees of  $\gamma$  and  $C$ .*

Furthermore, all the above quantities are integral multiples of  $h^i$  for the corresponding  $i$ . If  $\deg \gamma = 0$  then all the above quantities are 0.

**Remark 2.3.** Statement (1) holds for other dimensional topological cycles too. However, by Lefschetz hyperplane theorem, it is only interesting when  $\gamma$  has dimension  $n$ . The degree, or  $h$ -degree, of  $\gamma$  is defined as follows. If  $\gamma$  is an odd dimensional topological cycle, then  $\deg(\gamma) = 0$ ; If  $\gamma$  is a topological cycle of dimension  $2m$  or an algebraic cycle of dimension  $m$ , then  $\deg(\gamma) = \gamma \cdot h^m$ .

**Corollary 2.4.** *Let  $\gamma \in H_n(X, \mathbb{Z})$ ,  $\mathbf{a} \in \text{CH}_1(X)$  and  $\mathbf{b} \in \text{CH}_l(X)$ . Then the following are true*

- (i)  $2 \deg(\mathbf{a})\gamma + \Psi(\Phi(\gamma) \cdot \Phi([\mathbf{a}]))$  is a homology class that comes from  $\mathbb{P}^{n+1}$ .
- (ii) If  $n - 2 \geq l \geq 2$ , then  $2 \deg(\mathbf{a})\mathbf{b} + \Psi(\Phi(\mathbf{a}) \cdot \Phi(\mathbf{b}))$  is a cycle class that comes from  $\mathbb{P}^{n+1}$ .
- (iii) If  $l = 1$ , then  $2 \deg(\mathbf{a})\mathbf{b} + 2 \deg(\mathbf{b})\mathbf{a} + \Psi(\Phi(\mathbf{a}) \cdot \Phi(\mathbf{b}))$  is a cycle class that comes from  $\mathbb{P}^{n+1}$ .

*Proof of Theorem 2.2.* (1) Let  $f : M \rightarrow X$  be a continuous map from an  $n$ -dimensional topological manifold  $M$  to  $X$ . First we assume  $C$  is general in the sense that (i)  $C$  does not meet  $f(M)$ ; (ii)  $\Delta_0 := \{t \in M : f(t) \in D_C\}$  is purely of expected real dimension  $\dim M - 2$ . For each pair of points  $(t, x) \in M \times C$  there is a unique (complex projective) line passing through  $f(t)$  and  $x$ . This gives a continuous map  $M \times C \rightarrow G(2, n+2)$ . We get the following diagram

$$(1) \quad \begin{array}{ccccc} D_1 \cup D_2 \cup D & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & G(1, 2, n+2) & \longrightarrow & \mathbb{P}^{n+1} \\ \downarrow \pi & & \downarrow & & \\ M \times C & \longrightarrow & G(2, n+2) & & \end{array}$$

where all the squares are fiber products;  $D_1$  and  $D_2$  are sections of  $\pi$  corresponding the points on  $f(M)$  and  $C$  respectively. Let  $\pi' : D \rightarrow M \times C$  be the restriction of  $\pi$  to  $D$ . Let  $\Delta \subset M \times C$  be the closed subset of points  $(t, x)$  such that the line through  $f(t)$  and  $x$  is contained in  $X$ . By assumption  $\dim \Delta = \dim \Delta_0$ . Then  $\pi'$  is one-to-one away from  $\Delta$  while over  $\Delta$  it is an  $S^2$ -bundle with trivial Euler class (note that by taking the point in which the line meets  $C$ , we have a cross section of the bundle). Namely, we have the following diagram

$$(2) \quad \begin{array}{ccccc} \mathbb{P}^1 = S^2 & \longrightarrow & E & \longrightarrow & D \\ & & \downarrow & & \downarrow \pi' \\ & & \Delta & \longrightarrow & M \times C \end{array}$$

Let  $g : D \rightarrow X$  be the natural map. Let  $h$  be the homology class of hyperplane.

**Lemma 2.5.** *The following are true.*

- (i)  $g_*[D]$  is a class on  $X$  that comes from  $\mathbb{P}^{n+1}$ ;
- (ii)  $g_*(h|_D)$  is a class on  $X$  that comes from  $\mathbb{P}^{n+1}$ ;
- (iii)  $h|_D + [E] = 2\pi'^*c_1(\mathcal{E})$ , where  $\mathcal{E}$  is the pull back of the canonical rank 2 quotient bundle on  $G(2, n+2)$ ;
- (iv)  $c_1(\mathcal{E}) = p_1^*(h|_M) + p_2^*(h|_C)$ , where  $p_1 : M \times C \rightarrow M$  and  $p_2 : M \times C \rightarrow C$  are the natural projections;
- (v)  $g_*(\pi'^*c_1(\mathcal{E})) = -e f_*[M]$ , modulo classes coming from  $\mathbb{P}^{n+1}$ .

*Proof of Lemma 2.5.* For (i), we first note that the class of  $[D_1] + [D_2] + [D]$  on  $X$  is the restriction of  $[P]$  viewed as a class on  $\mathbb{P}^{n+1}$ . However, by dimension reasons, the class of  $[D_1]$  and  $[D_2]$  are all zero on  $X$ . To prove (ii), we note that by projection formula, we have  $g_*(h|_D) = g_*[D] \cdot h$ . The identity in (iv) is by construction. To prove (iii), we first note that  $h|_D + [E]$  restricts to zero on fibers  $E_t$  of  $E \rightarrow \Delta$  since  $h \cdot E_t = 1$  and  $E \cdot E_t = -1$ . By Leray-Hirsch theorem, this implies that  $h|_D + [E] = \pi'^*\mathbf{a}$  where  $\mathbf{a}$  is a cohomology class on

$M \times C$ . Apply  $\pi'_*$  to the above equation, we get  $\mathfrak{a} = \pi'_*(h|_D)$ . Note that on  $P$ , we have  $D_1 + D_2 + D = 3h$ , where, by abuse of notation, we still use  $h$  to denote the pull back to the hyperplane class to  $P$ . Hence

$$(3) \quad h \cdot D_1 + h \cdot D_2 + h \cdot D = 3h^2$$

Since  $P$  is the projectivization of  $\mathcal{E}$ , we have  $h^2 - \pi^*c_1(\mathcal{E})h + \pi^*c_2(\mathcal{E}) = 0$ . Apply  $\pi_*$  to (3), we get

$$\begin{aligned} \pi_*(h \cdot D) + \pi_*(h \cdot D_1) + \pi_*(h \cdot D) &= 3\pi_*(h^2) \\ &= 3\pi_*(\pi^*c_1(\mathcal{E}) \cdot h - \pi^*c_2(\mathcal{E})) \\ &= 3c_1(\mathcal{E}) \cdot \pi_*h = 3c_1(\mathcal{E}) \end{aligned}$$

We also easily get that  $\pi_*(h \cdot D_1) = p_1^*f^*h$  and  $\pi_*(h \cdot D_2) = p_2^*(h|_C)$ . Combine this with (iv), we get

$$\mathfrak{a} = \pi'_*(h|_D) = \pi_*(h \cdot D) = 2c_1(\mathcal{E}).$$

To prove (v), we note that for any class  $\mathfrak{a}$  on  $M \times C$ , if we pull back  $\mathfrak{a}$  to  $D_1 + D_2 + D$  and then push forward to  $X$ , then we get the same class as we pull back  $\mathfrak{a}$  to  $P$ , then push forward to  $\mathbb{P}^{n+1}$  and then restrict to  $X$ . As a result, we always get a class coming from  $\mathbb{P}^{n+1}$ . It follows that modulo classes coming from  $\mathbb{P}^{n+1}$ , we have the following equalities.

$$\begin{aligned} g_*(\pi'^*c_1(\mathcal{E})) &= -(g_{1*}(\pi^*c_1(\mathcal{E})|_{D_1}) + g_{2*}(\pi^*c_1(\mathcal{E})|_{D_2})) \\ &= -g_{1*}(\pi^*c_1(\mathcal{E})|_{D_1}) \\ &= -e f_*[M] \end{aligned}$$

Here  $g_i : D_i \rightarrow X$  is the natural map. This finishes the proof of the lemma.  $\square$

In the above lemma, we apply  $g_*$  to (iii) and then take (ii) and (v) into account. Note that  $g_*[E] = \Psi(\Phi(f_*[M]) \cdot F_C)$ . This way, we easily deduce the conclusion (1) of the theorem for  $\gamma = f_*[M]$ . But a general  $\gamma$  is a linear combination of  $f_*[M]$ 's, we get the conclusion by linearity.

we still need to remove the additional assumption on  $C$ . For this we use a specialization argument. Let  $\mathcal{M}$  be a component of the Hilbert scheme of  $X$  that contains  $[C]$  as a smooth point. If the conclusion (1) holds for a general point of  $\mathcal{M}$ , then by continuity it also holds for  $C$ . Now given any curve  $C$ , we attach sufficiently many very free rational curves  $C_i$  such that: (a) the resulting curve  $C'$  smoothes to a curve for which the conclusion holds; (b) the conclusion holds for all  $C_i$ 's. Then by taking limits we see that the conclusion holds for  $C'$  and hence also for  $C$  by linearity.

Now we start to prove (2) and (3) of Theorem 2.2. The basic strategy is the same as above. The only difference is that we need to consider the more delicate rational equivalence rather than homological equivalence. Let  $f : M \rightarrow X$  be a morphism from a smooth projective variety  $M$  to  $X$  such that  $f$  is birational onto its image. For this  $M$ , we pick a general  $C$  such that  $D_C$  intersects  $f(M)$  properly while  $C$  does not meet  $f(M)$ . As before we construct the natural morphism  $\varphi : M \times C \rightarrow G(2, n+2)$ . Let  $\mathcal{E}$  be the pull back of the canonical rank 2

quotient bundle to  $M \times C$ . In this way, we get the following diagram as before.

$$(4) \quad \begin{array}{ccccc} D_1 \cup D_2 \cup D & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & G(1, 2, n+2) & \longrightarrow & \mathbb{P}^{n+1} \\ \downarrow \pi & & \downarrow & & \\ M \times C & \longrightarrow & G(2, n+2) & & \end{array}$$

Here all the squares are fiber products;  $D_1$  and  $D_2$  are sections of  $\pi$  and contract to  $f(M)$  and  $C$  respectively. Let  $\pi' : D \rightarrow M \times C$  be the restriction of  $\pi$  to  $D$ . Then  $\pi'$  is a birational morphism. Let  $\Delta \subset M \times C$  be the subvariety that consists of points  $(t, x)$  such that the line through  $f(t)$  and  $x$  is contained in  $X$ . By assumption  $\Delta$  is generically smooth. Let  $\Delta^{sing}$  be the singular locus of  $\Delta$ . Then away from  $\Delta^{sing}$ ,  $\pi'$  is blow-up along  $\Delta$ . Let  $E$  be the exceptional locus of  $\pi'$ . We also know that the possible singularities of  $D$  can only appear over  $\Delta^{sing}$ . Take a resolution of singularities  $r : \tilde{D} \rightarrow D$ . Let  $E_i$  be the exceptional divisors of  $r$ . We use  $E'$  to denote the strict transform of  $E$  in  $\tilde{D}$ . We still use  $h$  to denote the hyperplane class induced from  $\mathbb{P}^{n+1}$ . Let  $g : \tilde{D} \rightarrow X$  be the natural morphism.

**Lemma 2.6.** *The following are true.*

- (i)  $g_*\tilde{D}$  is a class coming from  $\mathbb{P}^{n+1}$ ;
- (ii)  $g_*(h|_{\tilde{D}})$  is a class coming from  $\mathbb{P}^{n+1}$ ;
- (iii)  $h|_{\tilde{D}} + E' + \sum a_i E_i = 2(\pi' \circ r)^*c_1(\mathcal{E})$ , for some  $a_i \in \mathbb{Z}$ ;
- (iv)  $c_1(\mathcal{E}) = p_1^*(f^*h) + p_2^*(h|_C)$ ;
- (v) If  $\dim M \geq 2$ , then  $g_*(r^*\pi'^*c_1(\mathcal{E})) = -e f_*M$ , modulo classes coming from  $\mathbb{P}^{n+1}$ ;
- (vi) If  $M$  is a curve and  $e'$  is its degree, then  $g_*(r^*\pi'^*c_1(\mathcal{E})) = -e f_*M - e'C$ , modulo classes coming from  $\mathbb{P}^{n+1}$ .

*Proof of Lemma 2.6.* The proof goes in a similar way as that of Lemma 2.5. We note that the push forward of  $D_1 + D_2 + D$  to  $X$  is a class coming from  $\mathbb{P}^{n+1}$  because it is just the restriction of the class of the image of  $P$  in  $\mathbb{P}^{n+1}$ . Since  $D_1$  and  $D_2$  are contracted to smaller dimension, we get (i). Conclusion (ii) follows from the projection formula as before. (iv) is standard by construction. For (iii), note that on  $D - \pi'^{-1}(\Delta^{sing})$ , the class of  $h - E$  comes from a class  $\mathfrak{a}$  on  $M \times C - \Delta^{sing}$  via pull back by  $\pi'$ . Since the codimension of  $\Delta^{sing}$  in  $M \times C$  is at least 3, we know that the divisor class group of  $M \times C - \Delta^{sing}$  is the same as that of  $M \times C$ . If we look at the strict transforms on  $\tilde{D}$  of all these divisors, we see that they only differ by a divisor supported on the exceptional locus of  $r$ . Hence we easily get the equality in (iii). By a very similar argument as before we know that the class  $\mathfrak{a}$  is  $2c_1(\mathcal{E})$ . Conclusion (v) can be proved in the same way as that of Lemma 2.5. Conclusion (vi) is proved in the same way, only that we get an extra term in the last step because of the dimension reason.  $\square$

Now we come back to the proof of the theorem. In the above lemma, we first apply  $g_*$  to (3) and note that all the  $E_i$  map to zero. We also easily see that  $g_*E = \Psi(\Phi(f_*M) \cdot F_C)$ . Combine all these with (v) (or (vi) in the case of 1-cycles), we get the conclusion (2) and (3) for  $\gamma = f_*M$ . By linearity, the conclusions hold for any given  $\gamma$ .

We still need to remove the assumptions on  $C$ . But this can be easily done by a specialization argument as before.  $\square$

## 3. THE ACTION OF INCIDENCE CORRESPONDENCE

Let  $C \subset X$  be a smooth rational curve of degree  $e$  on  $X$ . We also assume that  $C$  is general, meaning that it comes from a non-empty open of the corresponding component of Hilbert scheme.

**Lemma 3.1.** *When  $C$  is general enough,  $S_C$  is smooth of dimension  $n - 2$ .*

*Proof.* This holds true by Bertini theorem. When  $e \geq 2$ , it is clear. Assume that  $C = l$  is a general line. By Lemma 1.4 in [Iz],  $F_l$  is smooth away from the point  $[l]$ . Hence  $S_l$  can only possibly be singular at points on  $\tilde{l} = p^{-1}([l]) \subset S_l$ . Since a general fiber of  $q : P \rightarrow X$  is smooth, hence a general fiber of  $S_l \rightarrow l$  is smooth. Hence there are only finitely many singular fibers of  $S_l \rightarrow l$ . For a singular fiber  $F_x$ ,  $x \in l$ , of  $S_l \rightarrow l$  to impose a singularity on  $S_l$ , the point  $[l] \subset F_x$  has to be in the singular locus. This is an extra condition. Hence for general  $l$ , there is no such fiber  $F_x$ .  $\square$

If  $e \geq 2$ , then there exist  $\frac{5e(e-3)}{2} + 6$  pair of points  $(y_i, z_i)$  on  $S_C$  such that each pair maps to a single point  $x_i \in F_C$ , where  $x_i$  corresponds to a secant line of  $C$ . A secant line of  $C$  is a line that meets  $C$  in two points, see §3 of [Sh] for more details. The  $x_i$ 's are the only singular points of  $F_C$  and  $p|_{S_C} : S_C \rightarrow F_C$  is the normalization and also a desingularization. If  $C = l \subset X$  is a general line then  $S_l \rightarrow F_l$  is a small resolution of singularity when  $n \geq 5$ . Here  $F_l$  has isolated singularity at the point  $[l]$  if  $n \geq 5$ . When  $n = 4$ ,  $F_l$  is a smooth surface and  $S_l$  is the blow up of  $F_l$  at the point  $[l]$ . In any case, the exceptional locus of  $S_l \rightarrow F_l$  is identified with  $l$ .

Now we assume that  $e \geq 2$ . We have the natural incidence correspondence  $I_C$  on  $F_C$ . The incidence correspondence  $I_C$  can generically be described as

$$(5) \quad I_C : [l] \mapsto \sum_{i=1}^{5e-5} [l_i],$$

where  $l$  is a line meeting  $C$  and  $l_i$  are the secant lines of the pair  $(l, C)$ , see [Sh]. Roughly speaking, a secant line of  $(C_1, C_2)$  is a line that meets both  $C_1$  and  $C_2$ . Note it could happen that  $l_i = l$  for some  $i$ . This happens when  $T_{C,x}$  is pointing in the positive direction of  $\mathcal{N}_{l/X}$ , where  $x = l \cap C$ . It is easy to see that  $I_C$  also induces an incidence correspondence  $I'_C$  on  $S_C$ .

**Remark 3.2.** We say that a correspondence  $\Gamma \subset Y \times Y$  is generically defined by  $y \mapsto \sum y_i$  if  $\Gamma$  is the closure of the graph of this multi-valued map.

Note that on  $F$ , we have a natural polarization  $g$  given by the Plücker embedding of  $G(2, n+2)$ . It can also be written as  $g = \Phi(h^2)$ . We fix  $g|_{F_C}$  as the polarization of  $F_C$ , which also induces a divisor class  $g|_{S_C}$  on  $S_C$ . Note that  $S_C$  admits a natural morphism  $q_0 = q|_{S_C} : S_C \rightarrow C$ . This gives an extra class  $g' = q_0^*[pt]$  on  $S_C$ . Note that  $g'$  is never ample. Let  $\sigma$  denote the action of  $I'_C$  on either cohomology or Chow groups of  $S_C$ .

**Proposition 3.3.** *Let  $C \subset X$  and  $\sigma$  be as above. Then the following are true.*

(1) *Let  $\mathbf{a}$  be a topological cycle of odd dimension on  $S_C$ . Then*

$$\sigma(\mathbf{a}) = \Phi_C(\Psi_C(\mathbf{a})) + \mathbf{a}$$

*holds in cohomology.*

(2) *Let  $\mathbf{a}$  be a topological cycle of dimension  $2m$  or an algebraic cycle of dimension  $m$ , then in either cohomology or Chow groups we have*

$$\sigma(\mathbf{a}) = \Phi_C(\Psi_C(\mathbf{a})) + \mathbf{a} + \text{const.}$$



where the constant only depends on the intersection numbers  $\mathbf{a} \cdot g^{n-m-2}$  and  $\mathbf{a} \cdot g'g^{n-m-3}$ .

*Proof.* Let  $\{\mathcal{C}_t : t \in T\}$  be a 1-dimensional family of rational curves on  $X$  such that  $\mathcal{C}_0 = C$  for a special point  $0 \in T$ . Let  $S_t = S_{\mathcal{C}_t}$  and  $I_t \subset S_t \times S_C$ ,  $t \neq 0$ , be the natural incidence correspondence. To be more precise,  $I_t$  can be generically described as

$$I_t : [l] \mapsto \sum_{i=1}^{5e} [l_i],$$

where  $[l] \in S_t$  and  $l_i$  are the secant lines of the pair  $(l, C)$ . When  $t$  specializes to 0,  $l$  specializes to a line meeting  $C$ ; there are 5 of the secant lines that specialize to five lines,  $E_i$ ,  $i = 1, 2, \dots, 5$ , passing through  $l \cap C$ ; all the other secant lines specialize to secant lines of  $(l, C)$ . We have the following description of the  $E_i$ 's. Consider the infinitesimal deformation of  $C$  along  $T$ . This gives a direction  $v \in H^0(C, \mathcal{N}_{C/X})$ . By choosing  $T$  general enough, we may assume that  $v$  is nowhere vanishing on  $C$ . At  $x = l \cap C$ , the directions  $T_{l,x}$ ,  $T_{C,x}$  and  $v$  span a linear  $\mathbb{P}^2$  in  $\mathbb{P}(T_{X,x})$  (this happens for all but finitely many  $l \in S_C$ ). All lines through  $x$  that is contained in this  $\mathbb{P}^2$  is a (2,3) complete intersection. Hence there are 6 such lines in total. These lines are exactly  $E_1, \dots, E_5$  and  $l$ . If  $C$  itself is a line, then we can take  $E_1 = C$ . We set  $\Gamma_v \subset S_C \times S_C$  to be the correspondence  $[l] \mapsto [l] + \sum [E_i]$ . Note that  $\Gamma_v$ , as an element in  $\text{CH}_{n-2}(S_C \times S_C)$ , is independent of the choice of  $v$  since the parameter space for  $v$  is  $\mathbb{P}(H^0(C, \mathcal{N}_{C/X}))$ .

Let  $I_0 = \lim_{t \rightarrow 0} I_t$ , see [Fu]. Then we have

$$(6) \quad I_0 = I'_C + \Gamma_v - \Delta_{S_C}$$

Let  $I_{F,C} \subset F \times S_C$  be the natural incidence correspondence. Then  $I_t = (I_{F,C})|_{S_t \times S_C}$  for  $t \neq 0$ . By taking limits, we know that  $I_0$  is  $I_{F,C}$  restricted to  $S_C \times S_C$ . Note that the action of  $I_{F,C}$  is  $\Phi_C \circ \Psi$ . Hence the action of  $I_0$  is equal to  $\Phi_C \circ \Psi_C$ . Then the proposition follows easily from the following lemma.  $\square$

**Lemma 3.4.** *Let  $\Gamma_v$  act either on cohomology or Chow groups of  $S_C$ .*

- (i) *If  $\mathbf{a}$  is an odd dimensional topological cycle, then  $\Gamma_v(\mathbf{a}) = 0$ ;*
- (ii) *If  $\mathbf{a}$  is zero dimensional, then  $\Gamma_v(\mathbf{a})$  is a constant that only depends on the degree of  $\mathbf{a}$ ;*
- (iii) *If  $\mathbf{a} = [S_C]$ , then  $\Gamma_v(\mathbf{a}) = 6\mathbf{a}$ ;*
- (iv) *If  $\mathbf{a}$  is a codimension  $2m$  topological cycle or codimension  $m$  algebraic cycle, then  $\Gamma_v(\mathbf{a})$  is a linear combination of  $g^m$  and  $g'g^{m-1}$  which only depends on the intersection numbers  $\mathbf{a} \cdot g^{n-m-2}$  and  $\mathbf{a} \cdot g'g^{n-m-3}$ .*

*Proof.* Let  $v \in H^0(C, \mathcal{N}_{C/X})$  be general. Then  $v$  defines a rank 2 sub-bundle  $\mathcal{T}_v \subset T_X|_C$ . At each point  $x \in C$ ,  $\mathcal{T}_{v,x}$  is the span of  $T_{C,x}$  and  $v(x)$ . Note that we have a natural embedding

$$S_C \hookrightarrow \mathbb{P}(T_X|_C) \longleftarrow \mathbb{P}(\mathcal{T}_v)$$

Note that  $S_C$  is a fiberwise (2,3) complete intersection of  $\mathbb{P}(T_X|_C)$ . If  $v$  is general, then  $S_C$  meets  $\mathbb{P}(\mathcal{T}_v)$  in a set  $\Sigma_v$  of finitely many points. Take  $\mathcal{F} = (T_X|_C)/\mathcal{T}_v$  to be the quotient bundle. Note that  $\Gamma_v$  actually defines an equivalence relation on  $S_C - \Sigma_v$ , i.e.  $l \sim l'$  if  $[l'] \in \Gamma_v([l])$ . Note that the equivalence class of  $[l]$  is given by the linear  $\mathbb{P}^2 \subset \mathbb{P}(T_{X,x})$  spanned by  $T_{l,x}$  and  $\mathcal{T}_{v,x}$ . This shows that we have the morphism

$$\rho : S_C - \Sigma_v \rightarrow \mathbb{P}(\mathcal{F}),$$

which is the quotient by the above equivalence relation. The action of  $\Gamma_v$  can be described as  $\rho^* \rho_*$ . The push forward  $\rho_*$  makes sense if the codimension of the cycle is at least 1. Let  $\xi$  be

the relative  $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$  class on  $\mathbb{P}(\mathcal{F})$ . Then the cohomology of  $\mathbb{P}(\mathcal{F})$  is naturally given by

$$H^*(\mathbb{P}(\mathcal{F})) = H^*(C)[\xi], \quad \xi^{n-2} + \pi^* c_1(\mathcal{F}) \cdot \xi^{n-3} = 0.$$

Where  $\pi : \mathbb{P}(\mathcal{F}) \rightarrow C$  is the natural projection. Similarly, we have the description of the Chow ring as

$$CH^*(\mathbb{P}(\mathcal{F})) = CH^*(C)[\xi], \quad \xi^{n-2} + \pi^* c_1(\mathcal{F}) \cdot \xi^{n-3} = 0.$$

Consider the following diagram

$$\begin{array}{ccccc} S_C & \xrightarrow{i} & \mathbb{P}(T_X|_C) & \xrightarrow{\alpha} & C \\ & \searrow \rho & \downarrow \beta & \nearrow \pi & \\ & & \mathbb{P}(\mathcal{F}) & & \end{array}$$

The rational map  $\beta$  is defined by the natural homomorphism  $T_X|_C \rightarrow \mathcal{F}$ . The locus where  $\beta$  is not defined is exactly  $\mathbb{P}(\mathcal{T}_v)$ . The pull back  $\beta^*\xi$  restricts to a hyperplane class on each fiber of  $\alpha$ . Hence  $\beta^*\xi$  differs from  $\mathcal{O}_{\mathbb{P}(T_X|_C)}(1)$  by a class from  $C$ . One also sees that  $\Phi(h^2)$  restricts to a hyperplane class of  $\mathbb{P}(T_{X,x}) = \alpha^{-1}(x)$ . Hence  $i^*\mathcal{O}_{\mathbb{P}(T_X|_C)}(1) = g|_{S_C}$  modulo a multiple of  $g' = i^*\alpha^*[pt]$ . It follows that  $\rho^*\xi = g + rg'$  for some integer  $r$ . Let  $\mathbf{a} \in H^{2m+1}(S_C)$  be a topological class of odd dimension, then  $\rho_*\mathbf{a} \in H^{2m+1}(\mathbb{P}(\mathcal{F})) = 0$ . Hence  $\Gamma_v(\mathbf{a}) = \rho^*\rho_*\mathbf{a} = 0$ . Now let  $\mathbf{a}$  be an element of  $H^{2m}(S_C)$  (or  $CH^m(S_C)$ ),  $0 < m < n-2$  (the cases of  $m=0, n-2$  are quite easy to deal with). Then  $\rho_*\mathbf{a}$  is an element in  $H^{2m}(\mathbb{P}(\mathcal{F}))$  (or  $CH^m(\mathbb{P}(\mathcal{F}))$ ). Then we have the following expression

$$\rho_*\mathbf{a} = a\xi^m + b\pi^*[pt] \cdot \xi^{m-1},$$

for some  $a, b \in \mathbb{Z}$ . Apply  $\rho^*$  to the above identity, we get

$$\Gamma_v(\mathbf{a}) = \rho^*\rho_*\mathbf{a} = a(g + rg')^m + bg'(g + rg')^{m-1} = ag^m + (b + ma)g^{m-1}g'$$

Also, the numbers  $a$  and  $b$  can be determined in the following way

$$\begin{aligned} a &= (a\xi^m + b\pi^*[pt]\xi^{m-1}) \cdot \pi^*[pt]\xi^{n-m-3} \\ &= \rho_*\mathbf{a} \cdot \pi^*[pt]\xi^{n-m-3} \\ &= \mathbf{a} \cdot g'(g + rg')^{n-m-3} \\ &= \mathbf{a} \cdot g'g^{n-m-3} \end{aligned}$$

To get  $b$ , we consider

$$\begin{aligned} \rho_*\mathbf{a} \cdot \xi^{n-m-2} &= a\xi^{n-2} + b \\ &= a(-\pi^* c_1(\mathcal{F}))\xi^{n-3} + b \\ &= -a \deg \mathcal{F} + b \end{aligned}$$

Hence we have

$$\begin{aligned} b &= a \deg(\mathcal{F}) + \mathbf{a} \cdot (g + rg')^{n-m-2} \\ &= \mathbf{a} \cdot g^{n-m-2} + ((n-1)e - 2 + r(n-m-2))\mathbf{a} \cdot g'g^{n-m-3} \end{aligned}$$

□

**Corollary 3.5** (of proof of Lemma). *The following identities hold.*

- (i)  $\Gamma_v(g'g^{m-1}) = 6g'g^{m-1}$ ;
- (ii)  $\Gamma_v(g^m) = 6g^m$ ;

*Proof.* Note that  $\rho^*\xi = g + rg'$  and  $g' = \rho^*(\pi^*[pt])$ . Hence both  $g$  and  $g'$  are pull-back via  $\rho$ . This implies that on the subring generated by  $g$  and  $g'$ , the action of  $\Gamma_v = \rho^*\rho_*$  is multiplication by  $\deg(\rho) = 6$ .  $\square$

**Lemma 3.6.**  $\Psi_C(g' \cdot \Phi_C(h^m)) = 2h^{m+1}$ .

*Proof.* First we note that  $\Psi_C(g' \cdot \Phi_C(h^m)) = \Psi(F_x \cdot \Phi(h^m))$ . Let  $M \subset X$  be a general complete intersection of hyperplanes that represents  $h^m$ . Choose  $x \in X$  to be a general point. Consider the following diagram

$$\begin{array}{ccccc} D_1 + D_2 + D & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & G(1, 2, n+2) & \longrightarrow & \mathbb{P}^{n+1} \\ \downarrow \pi & & \downarrow & & \\ M & \xrightarrow{\varphi} & G(2, n+2) & & \end{array}$$

where all the squares are fiber products;  $\varphi(x')$  is the line through  $x$  and  $x'$ ;  $D_1$  contracts to  $x$  and  $D_2$  maps isomorphically to  $M$ . Let  $\Delta \subset M$  be the intersection of  $M$  and  $\Psi(F_x)$ . Then  $\Delta$  is smooth of codimension 2 in  $M$ . The natural map  $D \rightarrow M$  is the blow up along  $\Delta$ . Let  $E \subset D$  be the exceptional divisor. As before, we have  $h|_D + E = 2(\pi|_D)^*(h|_M)$ . Push forward these classes to  $X$  and note that  $\pi^*(h|_M) \cdot D$  goes to  $3h^{m+1} - h \cdot M = 2h^{m+1}$  (here we use the fact that  $\pi^*(h|_M) \cdot (D_1 + D_2 + D)$  goes to  $3h^{m+1}$ ). Hence  $(\pi|_D)^*(h|_M)$  goes to  $2h^{m+1}$ . Similarly  $h|_D$  also goes to  $2h^{m+1}$  since  $D$  goes to  $2h^m$ . Hence the image of  $E$ , which represents the class  $\Psi(F_x \cdot \Phi(h^m))$ , is  $2h^{m+1}$ .  $\square$

Schubert calculus on  $G = G(2, n+2)$  shows that  $\text{CH}^m(G) = H^{2m}(G)$  is generated by Schubert varieties that are defined by flags  $\mathbb{P}^a \subset \mathbb{P}^b \subset \mathbb{P}^{n+1}$  with  $a < b \leq n+1$  and  $a+b = 2n+1-m$ . In our case, the Schubert varieties are lines that meet  $\mathbb{P}^a$  and are contained in  $\mathbb{P}^b$ . We refer to §14.7 of [Fu] for the details on Schubert calculus. Most of these classes restricts to 0 on  $S_C$  unless  $b = n$  or  $b = n+1$ . If  $b = n+1$  and  $a = n-m$ , then the corresponding Schubert class restricts to  $\Phi(h^{m+1})$  on  $F$  and hence  $\Phi_C(h^{m+1})$  on  $S_C$ . If  $b = n$  and  $a = n-m+1$ , then the corresponding Schubert class restricts to  $\deg(C)g' \cdot \Phi(h^m)|_{S_C}$  on  $S_C$ .

**Proposition 3.7.** (i)  $g^m$  is a linear combination of  $\Phi_C(h^{m+1})$  and  $g' \Phi_C(h^m)$ ;  $g'g^{m-1}$  is a multiple of  $g' \Phi_C(h^m)$ .

(ii) The action of  $\sigma$  preserves the algebra  $\mathbb{Q}[g, g']$ .

(iii) Under  $\Psi_C$ , the image of  $\mathbb{Q}[g, g']$  is  $\mathbb{Q}[h]_{\deg < n}$  where  $h$  is the hyperplane class of  $X$ .

*Proof.* We prove (i) by induction. Since  $g^m$  comes from  $G$ , we know that it can be written as a linear combination of  $\Phi_C(h^{m+1})$  and  $g' \Phi_C(h^m)$ . Hence we also see that  $g'g^{m-1}$  is a multiple of  $g' \Phi_C(h^m)$ . To prove (ii), we only need to consider the action of  $\sigma$  on  $\Phi_C(h^{m+1})$  and  $g' \Phi_C(h^m)$ . By the equation (6), we get

$$\begin{aligned} \sigma(\Phi_C(h^{m+1})) &= I_0(\Phi_C(h^{m+1})) - \Gamma_v(\Phi_C(h^{m+1})) + \Delta_{S_C}(\Phi_C(h^{m+1})) \\ &= \Phi_C \circ \Psi(\Phi(h^{m+1})|_{S_C}) - 6\Phi_C(h^{m+1}) + \Phi_C(h^{m+1}) \\ &= \Phi_C(a h^{m+1}) - 5\Phi_C(h^{m+1}), \text{ for some } a \in \mathbb{Z} \\ &= (a-5)\Phi_C(h^{m+1}) \end{aligned}$$

and

$$\begin{aligned}\sigma(g' \cdot \Phi(h^m)) &= \Phi_C \circ \Psi_C(g' \cdot \Phi_C(h^m)) - 5g' \Phi_C(h^m) \\ &= 2\Phi_C(h^{m+1}) - 5g' \Phi_C(h^m).\end{aligned}$$

To prove (iii), we also consider the action of  $\Psi_C$  on  $\Phi_C(h^{m+1})$  and  $g' \Phi_C(h^m)$ . By Theorem 2.2 and Lemma 3.6, we know that the images are multiples of  $h^{m+1}$ .  $\square$

#### 4. THE QUADRATIC RELATION AND PRYM-TJURIN CONSTRUCTION

Let  $C \subset X$  be a general smooth rational curve on  $X$ . We will use notations of the previous section. Let  $S_C = q^{-1}(C)$  as in the previous section. We will use notations from the previous section.

**Definition 4.1.** We define the *primitive cohomology*,  $H^*(S_C, \mathbb{Z})^\circ$ , of  $S_C$ , to be

$$\{\alpha \in H^*(S_C, \mathbb{Z}) : \alpha \cup \beta = 0, \forall \beta \in \mathbb{Q}[g, g']\}.$$

We define the *primitive Chow group*,  $\text{CH}^*(S_C)^\circ$ , of  $S_C$ , to be

$$\{\alpha \in \text{CH}^*(S_C) : \alpha \cdot \beta =_{\text{num}} 0, \forall \beta \in \mathbb{Q}[g, g']\}.$$

An element  $\mathfrak{a} \in H^*(S_C)^\circ$  is called a *primitive cohomology class*; An element  $\mathfrak{a} \in \text{CH}^*(S_C)^\circ$  is called a *primitive cycle class*.

By Proposition 3.7, the action  $\sigma$  induces an action, still denoted by  $\sigma$ , on both primitive cohomology and primitive Chow groups of  $S_C$ . This is because  $\sigma$  is symmetric and preserves the classes in  $\mathbb{Q}[g, g']$ . Hence  $\alpha \cdot \sigma(\beta) = \sigma(\alpha) \cdot \beta$ . If  $\alpha$  is a primitive class and  $\beta \in \mathbb{Q}[g, g']$ , then the above identity shows that  $\sigma(\alpha) \cdot \beta = 0$  and hence  $\sigma(\alpha)$  is still primitive.

**Theorem 4.2.** *Let  $C \subset X$  be as above. Let  $\sigma$  be the action of the incidence correspondence on either  $H^*(S_C)^\circ$  or  $\text{CH}^*(S_C)^\circ$ . Then the following are true.*

(1) *On the primitive part of either cohomology or Chow groups,  $\sigma$  satisfies the following quadratic relation*

$$(\sigma - 1)(\sigma + 2e - 1) = 0$$

(2) *The map  $\Phi_C$  induces an isomorphism of Hodge structures*

$$\Phi_C : H^n(X, \mathbb{Z})_{\text{prim}} \rightarrow P(H^{n-2}(S_C, \mathbb{Z})^\circ, \sigma)(-1)$$

*The intersection forms are related by the following identity*

$$\Phi_C(\alpha) \cdot \Phi_C(\beta) = -2e \alpha \cdot \beta$$

(3) *The map  $\Phi_C$  induces an isomorphism*

$$\Phi_C : A_i(X)_{\mathbb{Q}} \rightarrow P(\text{CH}_{i-1}(S_C)_{\mathbb{Q}}^\circ, \sigma)$$

*Proof.* Since  $\Gamma_v$  is also symmetric, we know that it acts on primitive cohomology and Chow groups. But by Lemma 3.4, the image of  $\Gamma_v$  is always non-primitive unless it is zero. Hence we get that  $\Gamma_v = 0$  on primitive cohomology and Chow groups. Hence on the primitive part of the cohomology and Chow groups, we have

$$\sigma = \Phi_C \circ \Psi_C + 1.$$

The next fact that we need is

**Lemma 4.3.** *If  $\mathfrak{a}$  is a primitive class in either cohomology or Chow group of  $S_C$ , then  $\Psi_C(\mathfrak{a})$  has  $h$ -degree equal to zero.*

The proof of the above lemma is easy. We note that  $\Psi_C(\mathbf{a}) \cdot h^i = \mathbf{a} \cdot \Phi_C(h^i) = 0$ . Now we can prove statement (1) of the theorem. On the primitive cohomology and primitive Chow group, we have

$$\begin{aligned} (\sigma - 1)(\sigma + 2e - 1)(\mathbf{a}) &= \Phi_C \circ \Psi_C(\Phi_C \circ \Psi_C + 2e)(\mathbf{a}) \\ &= \Phi_C(\Psi_C \circ \Phi_C(\Psi_C(\mathbf{a}))) + 2e\Phi_C \circ \Psi_C(\mathbf{a}) \\ &= \Phi_C(-2e\Psi_C(\mathbf{a})) + 2e\Phi_C \circ \Psi_C(\mathbf{a}), \quad (\text{by Theorem 2.2}) \\ &= 0 \end{aligned}$$

Now we prove (2). For simplicity, we write  $P$  for  $P(H^{n-2}(S_C, \mathbb{Z})^\circ, \sigma)$ . Since  $\Phi_C \circ \Psi_C = \sigma - 1$  and that  $\Psi_C$  is onto (Propositioon 4.4), we know that the image of  $\Phi_C$  is exactly  $P$ . By Theorem 2.2, we know that  $\Psi_C \circ \Phi_C = -2e$ . This implies that  $\Phi_C$  is injective. Hence  $\Phi_C : H^n(X, \mathbb{Z})_{\text{prim}} \rightarrow P$  is isomorphism.  $\Phi_C$  respects the Hodge structures and hence is an isomorphism of Hodge structures. Statement (3) can be proved exactly in the same way.  $\square$

**Proposition 4.4.** (1) *The homomorphism*

$$\Psi_C : H^{n-2}(S_C, \mathbb{Z})^\circ \rightarrow H^n(X, \mathbb{Z})_{\text{prim}}$$

*on primitive cohomology is surjective.*

(2) *The homomorphism*

$$\Psi_C : \text{CH}_m(S_C, \mathbb{Q})^\circ \rightarrow \text{A}_{m+1}(X, \mathbb{Q})$$

*on primitive Chow groups is surjective.*

*Proof.* Statement (1) follows from Theorem 5.1. The proof of (2) is easy since we have  $\mathbb{Q}$  coefficients. Let  $\alpha \in CH_{m+1}(X)_{\mathbb{Q}}$ , take  $\mathbf{a} = -\frac{1}{2e}\Phi_C(\alpha)$ . Then we have  $\alpha = \Psi_C(\mathbf{a})$ . Now assume that  $\mathbf{a}$  has degree 0. For any  $\mathbf{b} \in \mathbb{Q}[g, g']$ , we have  $\Phi_C(\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot \Psi_C(\mathbf{b}) = 0$ . This means that  $\Phi_C(\mathbf{a})$  is a primitive cycle class. This gives the surjectivity of  $\Psi_C$  on primitive Chow groups.  $\square$

Let  $\pi : \mathcal{C} \rightarrow T$  be a family of curves together with a morphism  $f : \mathcal{C} \rightarrow X$ . Then we can define the Abel-Jacobi homomorphism  $\Phi_T := \pi_* f^*$  on cohomology and the cylinder homomorphism  $\Psi_T := f_* \pi^*$  on Chow groups. Let  $C \subset X$  be a general smooth rational curve as before. Then there is a natural incidence correspondence  $\Gamma_{T,C} \subset T \times S_C$  given by

$$t \mapsto \sum [l_i]$$

where  $l_i$  are the secant lines of  $\mathcal{C}_t$  and  $C$ . This correspondence induces

$$(\Gamma_{T,C})_* : \text{CH}_r(T) \rightarrow \text{CH}_r(S_C)$$

and

$$[\Gamma_{T,C}]^* : H^{n-2}(S_C, \mathbb{Z}) \rightarrow H^{n-2}(T, \mathbb{Z})$$

We can define the primitive cohomology and Chow groups of  $T$  as the classes that are orthogonal to  $\Phi_T(\mathbb{Z}[h])$ . Hence  $\Gamma_{T,C}$  also induces homomorphism between primitive cohomology and Chow groups.

**Proposition 4.5.** *Let  $\mathcal{C} \rightarrow T$  be a family of curves on  $X$  and  $C$  be a general rational curve as above. All the homomorphisms are restricted to primitive classes. Then the following holds.*

(1) *If we identify  $H^n(X, \mathbb{Z})_{\text{prim}}$  with  $P(H^{n-2}(S_C, \mathbb{Z})^\circ, \sigma)$  via  $\Phi_C$ , then  $[\Gamma_{T,C}]^*$  is identified with  $\Phi_T$ .*

(2) *If we identify  $\text{A}_{r+1}(X, \mathbb{Q})$  with  $P(\text{CH}(S_C, \mathbb{Q})^\circ, \sigma)$  via  $\Phi_C$ , then  $(\Gamma_{T,C})_*$  is identified with  $\Psi_T$ .*

*Proof.* These are consequences of the identities  $[\Gamma_{T,C}]^* = \Phi_T \circ \Psi_C$  and  $(\Gamma_{T,C})_* = \Phi_C \circ \Psi_T$ .  $\square$

Let  $C_1$  and  $C_2$  be two general rational curves on  $X$  of degree  $e_1$  and  $e_2$  respectively. Then there is a natural incidence correspondence  $\Gamma_{12} \subset S_{C_1} \times S_{C_2}$  defined by

$$\Gamma_{12}([l]) = \sum_{i=1}^{5e_2} [l_i]$$

where  $[l] \in S_{C_1}$  and  $l_i$  are the secant lines of  $(l, C_2)$ . Let  $\gamma_{12}$  be the induced homomorphism on either cohomology or Chow groups. Note that by definition, we have  $\gamma_{12} = \Phi_{C_2} \circ \Psi_{C_1}$ . Since both  $\Phi_C$  and  $\Psi_C$  respects primitive classes, the above identity implies that  $\gamma_{12}$  takes primitive classes to primitive classes. We still use  $\gamma_{12}$  to denote the action on primitive cohomology or primitive Chow groups.

**Proposition 4.6.** *Let  $C_1, C_2 \subset X$  be two general rational curves and  $\gamma_{12}$  be the homomorphism induced by incidence correspondence as above. Let  $\sigma_1$  and  $\sigma_2$  be the action of incidence correspondence on  $S_{C_1}$  and  $S_{C_2}$  respectively. Then the following are true.*

(1) *The image of  $\gamma_{12} : H^{n-2}(S_{C_1}, \mathbb{Z})^\circ \rightarrow H^{n-2}(S_{C_2}, \mathbb{Z})^\circ$  is always in the Prym-Tjurin part. Furthermore there is an isomorphism of Hodge structures*

$$t_{12} : P(H^{n-2}(S_{C_1}, \mathbb{Z})^\circ, \sigma_1) \rightarrow P(H^{n-2}(S_{C_2}, \mathbb{Z})^\circ, \sigma_2)$$

*such that  $\Phi_{C_2} = t_{12} \circ \Phi_{C_1}$  and  $\gamma_{12} = t_{12}(\sigma_1 - 1)$ .*

(2) *The same conclusions as in (1) hold for primitive Chow groups with  $\mathbb{Q}$ -coefficient.*

*Proof.* For simplicity, we write  $\Lambda_i = H^{n-2}(S_{C_i}, \mathbb{Z})^\circ$  for the primitive cohomology and  $P_i = \text{Im}(\sigma_i - 1)$  for the Prym-Tjurin part,  $i = 1, 2$ . Let  $\Lambda = H^n(X, \mathbb{Z})_{\text{prim}}$ ,  $\Phi_i = \Phi_{C_i} : \Lambda \rightarrow P_i$ . Then one easily checks that  $t_{12} = \Phi_2 \circ \Phi_1^{-1}$  satisfies (1). The proof of (2) goes in the same way.  $\square$

## 5. SURJECTIVITY ON PRIMITIVE COHOMOLOGY

In this section we supply a proof of the surjectivity of  $\Psi_C$  on the primitive cohomologies. To do this, it is more convenient to consider homology instead of cohomology. We define

$$(7) \quad V_{n-2}(S_C, \mathbb{Z}) = \ker\{H_{n-2}(S_C, \mathbb{Z}) \rightarrow H_{n-2}(G, \mathbb{Z})\},$$

and

$$(8) \quad V_n(X, \mathbb{Z}) = \ker\{H_n(X, \mathbb{Z}) \rightarrow H_n(\mathbb{P}^{n+1}, \mathbb{Z})\}.$$

Under the Poincaré duality  $H_{n-2}(S_C, \mathbb{Z}) \cong H^{n-2}(S_C, \mathbb{Z})$ , the subspace  $V_{n-2}(S_C, \mathbb{Z})$  corresponds to  $H^{n-2}(S_C, \mathbb{Z})^\circ$ . And similarly we identify  $V_n(X, \mathbb{Z})$  with  $H^n(X, \mathbb{Z})_{\text{prim}}$ . Then the surjectivity of

$$\Psi_C : H^{n-2}(S_C, \mathbb{Z})^\circ \rightarrow H^n(X, \mathbb{Z})_{\text{prim}}$$

is equivalent to the following

**Theorem 5.1.** *The natural cylinder homomorphism*

$$\Psi_C : V_{n-2}(S_C, \mathbb{Z}) \rightarrow V_n(X, \mathbb{Z})$$

*is surjective.*

The idea of the proof is the Clemens-Letizia method, see [Cl] and [Le]. Our presentation closely follows that of [Shi]. Let  $\pi : V \rightarrow \Delta$  be a proper flat holomorphic map from a complex manifold  $V$  of dimension  $m + 1$  onto the unit disk  $\Delta$ . This map is called a *degeneration* if  $\pi$  is smooth over the punctured disk  $\Delta^* = \Delta - 0$  and  $V_t := \pi^{-1}(t)$  is irreducible for  $t \neq 0$ . Let  $\text{Sing}(V_0)$  denote the singular locus of  $V_0$ .

**Definition 5.2.** ([Shi]) A degeneration  $\pi : V \rightarrow \Delta$  is called *quadratic of codimension  $r$*  if  $\text{Sing}(V_0)$  is connected and, for every point  $p \in \text{Sing}(V_0)$ , there exist local coordinates  $(z_0, \dots, z_m)$  of  $V$  around  $p$  such that  $\pi = z_0^2 + \dots + z_r^2$ .

**Proposition 5.3.** *For any smooth cubic hypersurface  $X \subset \mathbb{P}^{n+1}$ , there exist a Lefschetz pencil  $X_t \subset \mathbb{P}^{n+1}$  with  $t \in B \cong \mathbb{P}^1$ , such that*

- (i) *The total space,  $\mathcal{F} = \cup_{t \in B} F(X_t)$ , of the associated Fano schemes of lines is smooth;*
- (ii)  *$X_0 = X$  is the cubic hypersurface we start with and  $X_t$  is smooth unless  $t \in \{t_1, \dots, t_N\}$ .*
- (iii) *For each degeneration point  $t_j \in B$ ,  $j = 1, \dots, N$ , the family  $\beta : \mathcal{F} \rightarrow B$  is a quadratic degeneration of codimension  $n - 2$  at the point  $t_j$ .*

This Proposition can be viewed as a special case of Proposition 1 of [Shi]. By the results of §7 of [CG], we easily get a description of the fibers of  $\beta$ . If  $t \in B - \{t_1, \dots, t_N\}$  then  $\mathcal{F}_t = F(X_t)$  is smooth of dimension  $2n - 4$ . Let  $x_i$  be the ordinary double point of  $X_{t_i}$  and  $\Gamma_i$  be the lines on  $X_{t_i}$  that pass through  $x_i$ . We know that  $\Gamma_i$  is smooth of dimension  $n - 2$ . The singular fiber  $\mathcal{F}_{t_i}$  is irreducible with  $\Gamma_i$  being the singular locus which is an ordinary double subvariety. Let  $z \in \Gamma_i$  be a point and  $Y \subset \mathcal{F}$  be an  $(n - 1)$ -dimensional complex submanifold is a small neighborhood of  $z$ . If  $Y$  meets  $\Gamma_i$  transversally in the point  $z$ , then  $z$  is a non-degenerate critical point of  $\beta|_Y$ . There is an associated vanishing cycle  $\sigma \in H_{n-2}(Y_{t_i+\epsilon}, \mathbb{Z})$ .

**Lemma 5.4.** *There exists a contractible analytic open neighborhood  $D$  of  $0 \in B$  containing  $\{t_1, \dots, t_N\}$  such that*

- (i) *There is an analytic family of curves  $\{C_t \subset X_t^{sm} : t \in D\}$  with  $C_0 = C$ , where  $X_t^{sm}$  is the smooth locus of  $X_t$ .*
- (ii)  *$\mathcal{S} = \cup_{t \in D} \mathcal{S}_{C_t}$  is a complex manifold and  $\rho : \mathcal{S} \rightarrow B$  is smooth away from the  $t_i$ 's;*
- (iii) *As sub-manifolds of  $\mathcal{F}$ ,  $\mathcal{S}$  meets each  $\Gamma_i$  transversally at finitely many points  $z_{i1}, \dots, z_{il}$ .*

*Proof.* Let  $\mathcal{H}/B$  be the component of relative Hilbert scheme of rational curves on  $X_t$ 's that contains  $C$  as an element. This is well-defined since  $C$  defines a smooth point in the relative Hilbert scheme. We claim that, after shrinking  $D$ , a general analytic section  $s : t \mapsto C_t \in \mathcal{H}$  satisfies  $C_t \subset X_t^{sm}$ . To prove this we first construct a free rational curve  $C_i \in X_{t_i}^{sm}$  that is of the same degree  $e$  as  $C$ . This can be done as follows. Let  $x_i$  be the ordinary double point of  $X_{t_i}$ . Consider the projection from the point  $x_i$ , we easily get the following diagram

$$\begin{array}{ccc} X_{t_i} & \xleftarrow{b} & \tilde{X}_{t_i} \\ & & \downarrow p \\ & & \mathbb{P}^n \end{array}$$

where  $b$  is the resolution by blowing up the double point and  $p$  is the projection from the double point. One checks that  $p$  is the blow up of  $\mathbb{P}^n$  along a subvariety  $Z \subset \mathbb{P}^n$  which is a  $(2, 3)$  complete intersection. A degree  $e$  rational curve in  $X_{t_i}^{sm}$  corresponds to a degree  $e$  rational curve that meets  $Z$  in  $2e$  points. An easy dimension count shows that such curve exists and a general one passes through a general point of  $X_{t_i}$ . Since  $C_i$  is free, it deforms to a curve  $C_t$  in nearby fibers. By a result of [CS], the space of degree  $e$  rational curves on a smooth

cubic hypersurface is irreducible. This implies that  $C_t$  is a point of  $\mathcal{H}$  and hence so is  $C_i$ . So we can choose a family  $s : t \mapsto C_t$  whose specializations does not meet the points  $x_i \in X_{t_i}$ . This proves (i). We also get (iii) easily by choosing  $s$  general enough. The smoothness of  $\mathcal{S}$  follows from a deformation argument. We only need to show that  $\mathcal{S}$  is smooth at the points  $z = z_{ij}$ . Let  $L$  be the line corresponds to  $z$  and  $C_i = C_{t_i}$ . Let  $v \in H^0(C_i, \mathcal{N}_{C_i/\mathbb{P}^{n+1}})$  be the section corresponding to  $\frac{\partial}{\partial t}$  at the point  $C_i$ . Let  $y = C_i \cap L$ . Then  $T_{\mathcal{S},z} \subset T_{\mathcal{F},z}$  is naturally given by all sections  $v' \in T_{\mathcal{F},z} \subset H^0(L, \mathcal{N}_{L/\mathbb{P}^{n+1}})$  with  $v'(y)$  pointing in the direction of  $v(y)$ . When  $v(y)$  is general, the above condition gives a codimension  $n - 2$  subspace of  $T_{\mathcal{F},z}$ . This proves (ii).  $\square$

**Remark 5.5.** From the above proof, we see that the family  $t \mapsto C_t$  can be made algebraic on some finite cover of  $B = \mathbb{P}^1$ . The points  $z_{ij}$  are exactly the critical points of  $\rho$  and all of them are non-degenerate.

Now we fix a small  $\epsilon$  and let  $B_i$  be the closed ball of radius  $\epsilon$  with center  $t_i$ . If  $\epsilon$  is small enough, we have  $B_i \subset D$ . Pick a path  $l_i$  connecting 0 and  $t_i + \epsilon$  such that  $\cup_i l_i$  is star-shaped and contractible. Let  $D_i \subset \mathbb{P}^{n+1}$  be a small open ball centered at the double point  $x_i \in X_{t_i}$ . Let  $\tilde{D}_i \subset \mathcal{F}$  be the set of lines  $L \in \mathcal{F}$  such that  $L$  meets  $D_i$ . Thus  $\tilde{D}_i$  is a small open neighborhood of  $\Gamma_i$ . By construction,

$$\mathcal{S} \cap \tilde{D}_i = U_1 \cup \cdots \cup U_l$$

where  $U_j$  are disjoint open balls in  $\mathcal{S}$ . Let  $p_j : P_j \rightarrow U_j$  be the family of lines parameterized by  $U_j$ . Hence we get the following commutative diagram

$$\begin{array}{ccc} P_j & \xrightarrow{q_j} & Y \\ p_j \downarrow & & \downarrow \pi \\ U_j & \xrightarrow{\rho_j} & B \end{array}$$

where  $Y$  is the blow up of  $\mathbb{P}^{n+1}$  along the base locus of the Lefschetz pencil and  $\rho_j = \rho|_{U_j}$ . We fix a general analytic section  $s_j : U_j \rightarrow P_j$ .

**Lemma 5.6.** *There exists local analytic coordinates  $u_0, u_1, \dots, u_n$  of  $\mathbb{P}^{n+1}$  at the point  $x_j$  such that*

- (i) *The image of  $P_j$  in  $\mathbb{P}^{n+1}$  is locally given by  $u_0 + \sqrt{-1}u_1 = 0$ ;*
- (ii) *The image  $q_j \circ s_j(U_j)$  is given by  $u_0 = u_1 = 0$ ;*
- (iii)  *$\pi = t_j + u_0^2 + u_1^2 + \cdots + u_n^2$ .*

*Proof.* We first show that  $q_j$  has injective tangent map. For any tangent vector  $v \in T_{U_j, z_j}$ , it corresponds to a global section of  $\mathcal{N}_{L_j/\mathbb{P}^{n+1}}$ , where  $L_j$  is the line corresponding to  $z_j$ . By the choice of  $C_t$ , we know that  $C_{t_i}$  meets the divisor swept out by lines through  $x_i$  transversally. Then  $v$  does not vanish at the point  $x_i \in L_j$ . This means that the point  $x_i \in L_j$  actually moves when we move  $L_j$  in  $U_j$ . Hence we get the injectivity of the tangent map of  $q_j$ . Then we can start with local coordinates on  $U_j$  and extend to  $P_j$  and then to  $\mathbb{P}^{n+1}$ . See Lemma 7 of [Shi].  $\square$

Since we are doing local computations, by abuse of notation, we regard  $P_j$  as a submanifold of  $\mathbb{P}^{n+1}$ . It is standard that a vanishing cycle associated to  $x_i$  is given by

$$\Sigma_{ij} = \{u_0^2 + \cdots + u_n^2 = \epsilon, \quad u_k \in \mathbb{R}\}$$



See [La] for more details. We fix an orientation of  $\Sigma_{ij}$  and this gives an element  $[\Sigma_{ij}] \in H_n(X_{t_i+\epsilon}, \mathbb{Z})$ . Also, by Lemma 5.6, we know that  $\sigma_{ij} = P_j \cap \Sigma_i$ , with the induced orientation, gives a vanishing cycle for the critical point  $z_{ij} \in \mathcal{S}$ . Let

$$[\sigma_{ij}] \in H_{n-2}(\mathcal{S}_{t_i+\epsilon}, \mathbb{Z})$$

be the corresponding homology class. Now if we fix the class  $[\Sigma_i]$  at the very beginning, then  $\Sigma_{ij}$  has an induce orientation and hence so does  $\sigma_{ij}$ .

**Proposition 5.7.** ([Shi]) *For any  $\mathbf{a} \in H_{n-2}(\mathcal{S}_{t_i+\epsilon}, \mathbb{Z})$ , we have*

$$\mathbf{a} \cdot ([\sigma_{i1}] + \cdots + [\sigma_{il}]) = \Psi(\mathbf{a}) \cdot [\Sigma_i]$$

where  $\Psi = \Psi_{C_{t_i}}$ .

*Proof.* The proof goes the same as that of Proposition 2 in [Shi]; we sketch it here for completeness. We still use  $\mathbf{a}$  to denote a topological cycle that represents the class  $\mathbf{a}$ . We use the notations above and set  $\mathbf{a}_j = \mathbf{a} \cap U_j$ . If  $L \in \mathbf{a} - \cup_j \mathbf{a}_j$ , then  $L \cap \Sigma_i = 0$ . We may assume that  $\mathbf{a}_j$  meets  $\sigma_{ij}$  transversally at  $\mu$  points  $a_1, \dots, a_\mu \in \sigma_{ij}$ . Then the intersection of  $\cup_{L \in \mathbf{a}_j} L$  and  $\Sigma_{ij}$  are exactly  $s_j(a_1), \dots, s_j(a_\mu)$ . And by construction, this intersection is transversal. By chasing the orientations, we see that the contributions to the two sides of the identity are equal.  $\square$

**Remark 5.8.** Since  $\cup_i l_i$  is contractible,  $H_n(X_{t_i+\epsilon}, \mathbb{Z})$  is naturally identified with  $H_n(X, \mathbb{Z})$  via  $(l_i)_*$ . We have similar identification for  $\mathcal{S}_{t_i+\epsilon}$  and  $S_C = S_0$ . Hence we can view  $[\sigma_{ij}]$  as elements in  $V_{n-2}(S_C, \mathbb{Z})$  and  $[\Sigma_i]$  as elements in  $V_n(X, \mathbb{Z})$ . Under this identification, the above proposition still holds true.

Now we are ready to prove the main result of this section.

*Proof.* (of Theorem 5.1). If  $n$  is odd then the primitive cohomology is the whole cohomology group. In this case the surjectivity was already shown in the proof of Proposition 4.4. Assume that  $n$  is even. By Lefschetz theory,  $V_n(X, \mathbb{Z})$  is generated by vanishing cycles, see [La]. Since  $\Psi = \Psi_C$  commutes with the specialization map, we know that  $\Psi([\sigma_{i1}]) = \lambda[\Sigma_i]$  for some  $\lambda \in \mathbb{Z}$ . Now we see that

$$[\sigma_{i1}] \cdot ([\sigma_{i1}] + \cdots + [\sigma_{il}]) = [\sigma_{i1}] \cdot [\sigma_{i1}] = \pm 2$$

See [La], p.40. By the above proposition, we get

$$\Psi([\sigma_{i1}]) \cdot [\Sigma_i] = \lambda[\Sigma_i] \cdot [\Sigma_i] = \pm 2$$

Comparing this with  $[\Sigma_i] \cdot [\Sigma_i] = \pm 2$ , we get  $\lambda = \pm 1$ . This shows that  $[\Sigma_i]$  is in the image of  $\Psi$ . This proves the theorem since  $[\Sigma_i]$  generates  $V_n(X, \mathbb{Z})$ .  $\square$

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